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A NEW CENTER OF GRAVITY FORMULA OF GENERAL APPLICABILITY.

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The formula

$$\tfrac{1}{2}D+\frac{(B-A)D^2}{12V}, \qquad (1)$$

wherein A is the area of the generatrix of a space, in its initial position, B, its area in its final position, D is the distance between these positions, and V is the amount of space between, is an expression representing a distance, or linear measurement, like in kind to D, because the second term is a product of four dimensions divided by a product of three dimensions of like kind. If the space be a plane, the second term is still, as well as the first, an expression of a single dimension. This expression has been found to represent the distance of the center of magnitude from the initial end, of each of a number of shapes, or the center of gravity of a mass of uniform density, which fills the space.*

To determine the extent of its applicability, let us first consider two spaces,

$$y=f_1(x)_0^D, \qquad y=f_2(x)_0^D. \qquad (2)$$

Let A_1, B_1, be the end areas of first, V_1, its volume, D, the distance between these limits, and D_1, the distance, from first end, of the center of its magnitude. Let A_2, B_2, V_2, D, D_2, be the similar values of the second space, both lying between same limiting planes, as indicated in (2).

If formula (1) apply to each of these spaces, then

$$D_1=\tfrac{1}{2}D+\frac{(B_1-A_1)D^2}{12V_1},$$

$$D_2=\tfrac{1}{2}D+\frac{(B_2-A_2)D^2}{12V_2}.$$

The distance, D_3, of the center of gravity of the two spaces from the initial plane, is

$$D_3=\frac{D_1V_1+D_2V_2}{V_1+V_2}=$$

$$\frac{\tfrac{1}{2}DV_1+\tfrac{1}{12}(B_1-A_1)D^2+\tfrac{1}{2}DV_2+\tfrac{1}{12}(B_2-A_2)D^2}{V_1+V_2}$$

$$=\tfrac{1}{2}D+\frac{(B_1+B_2-A_1-A_2)D^2}{12(V_1+V_2)}=\tfrac{1}{2}D+\frac{(B_3-A_3)D^2}{12V_3}$$

wherein A_3, B_3, V_3, are the areas of ends, and volume of combined space. Therefore, formula (1) applies to the space

$$y=[f_1(x)+f_2(x)]_0^D=f_3(x)_0^D.$$

In the same manner it may be shown that the formula applies to the space

$$y=[f_3(x)+f_4(x)]_0^D,$$

if it apply to space, $y=f_4(x)_0^D$, and so on. Consequently, *if formula* (1) *apply to each of any number of spaces between same limits, it applies to their sum.*

From a similar course of reasoning it follows that *If formula* (1) *apply to all but one of any number of spaces between same limits, it does not apply to their sum.* Also, *if the formula apply to certain spaces between same limits and do not apply to certain other spaces, more than one, between the same limits as before, the formula does not apply to the sum of all the spaces, except in special cases, when its error for some of the spaces is balanced by its error, with opposite sign, for the rest.*

Now let us consider the expression for

* Formulæ for R. R. Earthwork, Second Edition, p. 105.

varies as a quadratic function,

all spaces, $y=F(x)$.* Each term of this is of the form Kx^n; and the distance of the center of gravity, from the first end, of the space represented by $y=Kx^n$, is

$$\frac{\int_0^D Kx^{n+1}dx}{\int_0^D Kx^n\,dx}=\frac{n+1}{n+2}D.$$

Formula (1) applies to this space when

$$\tfrac{1}{2}D+\frac{\dfrac{(KD^n-K(0)^n)D^2}{12}}{\dfrac{12}{n+1}KD^{n+1}}-\frac{n+1}{n+2}D=0. \qquad (3)$$

This is true when $n=0$. For all other values of n, (3) becomes

$$\tfrac{1}{2}D+\frac{n+1}{12}D-\frac{n+1}{n+2}D=0. \qquad (4)$$

From this $n^2-3n+2=0$;

whence $n=\frac{3}{2}\pm\frac{1}{2}=2$ or 1.

It follows from this and the rules written in italics, that the center of gravity formula applies, between any limits, to those spaces only which are represented by the equation

$$y=a+bx+cx^2; \qquad (5)$$

and it applies to special cases only, [*i. e.*, between special limits,] of all other spaces.

By comparison of (5) with similar equation, [eq. (11), p. 412, May No. of this Magazine,] representing the limit of the prismoidal formula's applicability, which equation is

$$y=a+bx+cx^2+dx^3, \qquad (6)$$

it is seen that the center of gravity formula is, *practically*, co-extensive with the prismoidal formula; because the generatrices of very few, if any, practical shapes vary as cubic functions of the path. This formula will, therefore, serve in the same way as the prismoidal, as a widely general rule, which renders unnecessary the demonstration, recollection and use of a large number of special rules.

Thus, the chapter on center of gravity in a treatise on mechanics can be much abbreviated by use of this formula. The fact that the generatrix of a space, expressed in terms of its distance from initial end of that space, or from any position whatever, since to shift the axis of Y does not change the degree of eq. (5), is all that requires demonstration. The position of the center of magnitude in that space is, then, immediately indicated by formula (1), which may be reduced for particular cases.

For instances: Because the areas of generatrices of triangle, pyramid, cone, and paraboloid, in terms of distance from their vertices, are, respectively, ax, $b(cx)^2$, $\pi(ex)^2$ and πfx, formula (1) applies to all of these and to all of their frusta. Therefore, for triangle, the distance of center of gravity from vertex is

$$\tfrac{1}{2}D+\frac{(B-0)D^2}{12(\frac{1}{2}DB)}=\tfrac{2}{3}D.$$

For trapezoid;

$$\text{dist. c. of g.}=\tfrac{1}{2}D+\frac{(B-A)D^2}{12(B+A)\frac{1}{2}D}=\frac{(2B-A)D}{3(B+A)}.$$

For pyramid or cone when B is area of base,

$$\text{dist. c. of g.}=\tfrac{1}{2}D+\frac{BD^2}{12(\frac{1}{3}BD)}=\tfrac{3}{4}D.$$

For frustum of this:

$$\text{dist. c. of g.}=\tfrac{1}{2}D+\frac{(B-A)D^2}{12(A+\sqrt{AB}+B)\frac{1}{3}D}=$$

$$\tfrac{1}{2}D+\frac{(B-A)D}{4(A+\sqrt{AB}+B)}.$$

The last result, which is formula (1) only changed by cancelation of the common factor D, is at once in the simplest form. Dr. Weisbach requires a page and a quarter of laborious demonstration to reach this result. See pp. 233-4, Eckley Coxe's edition of *Theoretical Mechanics*.

For paraboloid, the radius of whose base is r,

$$\text{dist. c. of g.}=\tfrac{1}{2}D+\frac{\pi r^2D^2}{12\pi r^2\frac{1}{2}D}=\tfrac{2}{3}D.$$

For frustum of this we obtain an expression similar to that for trapezoid, since its generatrix varies according to a function exactly similar. It is

$$\text{dist. c. of g.}=\tfrac{1}{2}D+\frac{\pi(r_2^2-r_1^2)D^2}{12\pi(r_2^2+r_1^2)\frac{1}{2}D}=$$

$$\frac{(2r_2^2-r_1^2)D}{3(r_2^2+r_1^2)}$$

Formula (1) applies to the sphere, because, with center as origin, the magnitude of generating circle varies as

* See introduction to article entitled *Prismoidal Formula* in May number of this Magazine.

$$\pi(r^2-x^2).$$

Hence, for hemisphere, measured from center,

$$\text{dist. c. of g.}=\tfrac{1}{2}r-\frac{\pi r^4}{12(\frac{2}{3}\pi r^3)}=\tfrac{3}{8}r.$$

Because the generatrix of the complete rectangular prismoid [Fig. 2, p. 414, May No. of this Magazine] varies as

$$a+bx+cx^2,$$

formula (1) applies to all rectangular prismoids and wedges, in direction perpendicular to bases.

$$\therefore \text{ dist. c. of g.}=\tfrac{1}{2}\text{D}+\frac{(\text{B}-\text{A})\text{D}^2}{12(\text{A}+4\text{M}+\text{B})\frac{1}{6}\text{D}}=$$

$$\tfrac{1}{2}\text{D}\left\{1+\frac{\text{B}-\text{A}}{\text{A}+4\text{M}+\text{B}}\right\}=\frac{(\text{B}+2\text{M})\text{D}}{\text{A}+4\text{M}+\text{B}}.$$

For wedge, measured from edge;

$$\text{dist. c. of g.}=\tfrac{1}{2}\text{D}\left\{1+\frac{\text{B}}{\text{B}+4\text{M}}\right\}$$

$$=\frac{(\text{B}+2\text{M})\text{D}}{\text{B}+4\text{M}}.$$

To reach results similar to these for wedge and prismoid requires more than a page of Weisbach's Mechanics. By calculus, the demonstration is nearly as long.

In a manner similar and quite as easy it may be shown that formula (1) applies to any segment of common parabola, in direction perpendicular to axis, to all parallelograms, prisms, cylinders, prismoids, cylindroids, spheroids, hyperboloids, all segments of these between parallel planes, and to the class of shapes whose lateral boundaries are right line surfaces the type of which is illustrated in Fig. 6, p. 418, of May No. of this Magazine. The last class includes alone, columns, chimneys, piers, abutments, warped-faced wing-walls, banks, retaining walls, dams, earthwork solids, and numerous other practical shapes.

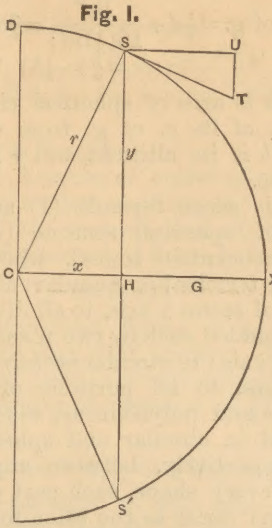
If the extremity, S, of the generatrix of a circle, Fig. 1, generate a length, dx, of the circumference in an infinitesimal time, dt, while it passes through center, then, at any distance, x, it will in same time generate a length equal to

$$dx.\frac{\text{ST}}{\text{SU}}=dx.\frac{r}{y}=dx.\frac{r}{\sqrt{r^2-x^2}}.$$

Hence formula (1) does not apply to arcs of circles.

But, if the circumference be revolved about CX, then the circumference of the generating circle whose radius is y, generates in the time, dt,

Fig. I.



$$dx.\frac{r}{y}.2\pi y=2\pi r dx,$$

which is the form of eq. (5), when $b=o$, $c=o$. Hence formula (1) applies to any zone of sphere, and the position of center of magnitude is, when h is altitude of zone,

$$\tfrac{1}{2}h+\frac{(2\pi r dx-2\pi r\, dx)h^2}{12\text{ V}}=\tfrac{1}{2}h.$$

Because formula (1) applies to the cone CSS′, and to the segment SXS′, as has already been shown, it may be used to determine the positions of the centers of gravity of these; and the c. of g. of sector may then be found by composition of moments.

Formula (1) applies directly to the spherical sector when D is understood to be the distance passed over by the center of magnitude of generatrix, a varying, concentric zone, and V to be the amount of space described by a plane generatrix, varying as the same function of, and passing over, the same path.

This is true, because the distribution of magnitude along the path D is identical for both modes of generation, and because, since formula (1) applies to one of these shapes, the variation of generatrix being as cx^2, it also applies to the other.

Thus is obtained for sector of sphere,

$$\text{dist. c. of g.} = \tfrac{1}{2}d + \frac{Bd^2}{12(\frac{1}{3}Bd)} = \tfrac{3}{4}d$$
$$= \tfrac{3}{4}(r - \tfrac{1}{2}h) \quad . \quad . \quad (7)$$

where B is area of spherical circle, d is distance of its c. of g. from center of sphere, h is its altitude, and r is radius of sphere.

In this sense formula (1) applies to frusta of spherical sectors, [*i. e.*, between concentric zones], whether the zones be terminal or annular; also, in direction of sector's axis, to all divisions of these included each by two planes passed through axis; to circular sectors between any limits, to all portions of regular polygons and polyhedrons, which can be inscribed in circular and spherical sectors, respectively, between any limits, and to every shape, each part of whose generatrix varies as the same function of its distance from initial end.

Thus, it applies to the square diagonally, when the initial end is one vertex, and the generatrix is the broken line composed of the opposite two sides.

Applied in this way, formula (1) generally requires, preliminarily, the aid of some other method to determine the c. of g. of generatrix itself. In case of spherical sector, it applies both to the generatrix and to the space described thereby.

It will be observed from the foregoing that the only familiar simple shapes, such as are made examples in works on mechanics and engineering, to which formula (1) does not apply *directly*, are the circular arc, sector, segment and spandrel; and to latter three of these it applies either indirectly or with aid of composition of moments.

Formula (1) applies to all lines for which

$$\sqrt{\frac{dy^2}{dx^2}+1} = a + bx + cx^2, \qquad (8)$$

where a, b, c, are arbitrary. If $c=o$, dy is always in integrable shape; but the integration is tedious. Eq. (8), however, shows at once that the only familiar line to which formula (1) applies is the straight line. To obtain another example, without much labor in the integration, make a and c zero and square, subtract unity, extract square root, and multiply by dx, each member of eq. (8). By integration the equation of a line, subject to formula (1) is found to be

$$y = \tfrac{1}{2}bx\sqrt{x^2 - \frac{1}{b^2}} - \frac{1}{2b}\text{ nap. log.}$$
$$\left\{ x + \sqrt{x^2 - \frac{1}{b^2}} \right\} + \text{const.}$$

Formula (1) applies in direction of axis to all surfaces formed by revolution of the lines for which

$$y\sqrt{\frac{dy^2}{dx^2}+1} = a + bx + cx^2, \qquad (9)$$

where a, b, c, are arbitrary. The only familiar surfaces of revolution for which eq. (9) is true are the surfaces of cylinder, cone and sphere.

Formula (1) applies, as a center of pressure formula, to the common cases of hydrostatic pressure. The equation of any immersed plane figure, referred to intersection of its plane with surface of fluid, as the axis of Y, and to any line in its plane, perpendicular to this intersection, as the axis of X, is

$$y = F(x).$$

If θ be the angle of deviation of its plane from plane of fluid surface, and γ be the specific gravity of the fluid, supposed to be uniform in density, the pressure on the figure at a distance x is

$$y' = F(x)\gamma \sin\theta\, x = F'(x).$$

In order that formula (1) shall apply,

$$F'(x) = a + bx + cx^2.$$

$$\therefore F(x) = \frac{a + bx + cx^2}{\gamma \sin\theta . x} = \frac{a'}{x} + b' + c'x. \ (10)$$

Fortunately, the last member of eq. (10), when we make $a'=o$, represents nearly all the usual submerged plane shapes, since these are seldom other than the rectangle, triangle and trapezoid, with bases parallel to fluid surface. Because the pressure at every point of submerged surface is normal thereto, and equal in intensity to γ sin θ x, it may be represented by a line perpendicular to surface and varying in length as the same constant multiple of x. Therefore, the equation of entire pressure,

$$y = (b' + c'x)\gamma \sin\theta . x = bx + cx^2,$$

is also the equation of a space, whose generatrix is a rectangle varying as y in last equation, its sides remaining constant in direction. This shape can, consequently, be only a frustum of the rectangular pyramid, prism or prismoid.

We may, therefore, confine our attention wholly to these shapes, because the resultant of pressure passes through the center of magnitude of each.

Of course, as shown by expression (6), the prismoidal formula also applies to these shapes, and, in consequence, to the pressure. Indeed, it would seem that this is the simplest method of determining the total pressure. The following is the

RULE.

To find the total pressure on a submerged rectangle, triangle or trapezoid, whose bases are parallel to fluid surface,

Multiply top-width of shape by its distance beneath surface; call this A. *Multiply bottom width by its distance beneath surface; call this* B. *Multiply mid-width by its distance beneath surface; call this* M. *Then total pressure is* $\frac{1}{6}$ *of distance between end-widths, multiplied by the specific gravity of fluid, and again by*

$$A+4M+B.$$

When top-width is w_1, and its distance beneath surface is h_1; and w_2, h_2, correspond to lower width, while D is distance between, the formula is

$$P=\tfrac{1}{6}D\gamma[w_1h_1+(w_1+w_2)\,(h_1+h_2)+w_2h_2] \quad . \quad . \quad (11)$$

If it be desired to determine the component of pressure in any direction, use, instead of D, the projection of D normal to that direction.

The formula determining the distance of center of pressure, measured from fluid surface along submerged plane, is

$$d_1+\tfrac{1}{2}D+\frac{(B-A)D^2\gamma}{12P}, \quad . \quad . \quad . \quad (12)$$

where d_1 is distance from fluid surface to top base, measured on plane of figure, and the other symbols are as before. (12) may be written

$$d_1+\tfrac{1}{2}D+\frac{(B-A)D}{2(B+4M+A)}, \quad (13)$$

or

$$d_1+\tfrac{1}{2}D+\frac{(w_2h_2-w_1h_1)D}{2[w_1h_1+(w_1+w_2)(h_1+h_2)+w_2h_2]} \quad . \quad . \quad . \quad (13a)$$

When, instead of h_1, h_2, we know the distances d_1, d_2, along submerged plane, formula (11) becomes

$$P=\tfrac{1}{6}D\gamma\sin\theta[w_1d_1+(w_1+w_2)(d_1+d_2)+w_2d_2] \quad . \quad . \quad . \quad (14)$$

and formula (13*a*) becomes

$$d_1+\tfrac{1}{2}D+\frac{(w_2d_2-w_1d_1)D}{2[w_1d_1+(w_1+w_2)(d_1+d_2)+w_2d_2]} \quad (15)$$

as simple as before.

To find distance of center of pressure below fluid surface, multiply (15) by $\sin\theta$, or use in (15), instead of d_1, d_2, D, their vertical projections, if these be known

EXAMPLES.

The submerged figure is a triangle whose base is at fluid surface. Here, d_1 and w_2 are zero; consequently, the distance of center of pressure is $\frac{1}{2}$D. The pressure is

$$\tfrac{1}{6}D\gamma\sin\theta\, w_1d_2 \text{ or } \tfrac{1}{6}D\,\gamma\, w_1h_2.$$

This corresponds to the case of the entire middle branch of the complete prismoid, Fig. 2, p. 414, May No. of this Magazine.

The vertex of the triangle is at surface, and the base is parallel thereto. Here w_1 and d_1 are zero; and, in consequence, formula (15) reduces to $\frac{3}{4}$ D. Then,

$$P=\tfrac{1}{3}D\gamma\sin\theta.\, w_2d_2, \text{ or } \tfrac{1}{3}D\gamma\, w_2h_2.$$

The figure is a rectangle. Here $w_1=w_2$, and the formula (15) becomes

$$d_1+\tfrac{1}{2}D+\frac{(d_2-d_1)D}{6(d_2+d_1)}.$$

$$P=\tfrac{1}{2}D\gamma\sin\theta.\, w_1(d_1+d_2).$$

The distribution of pressure on submerged trapezoids corresponds to the distribution of magnitude in the various segments of the complete rectangular prismoid, as wedges, etc.

If the plane of submerged figure be parallel to fluid surface, formula (13*a*) shows that the center of pressure is coincident with the center of magnitude of the plane shape itself. When, now, $w_1=w_2$, we have the case corresponding to that of the rectangular prism.

While defining the rectangular prismoid on p. 414, May number of this magazine, as a shape generated by a moving rectangle, the product and quotient of whose two dimensions vary, we noticed two other shapes, in one of which—the pyramid—the product is variable and the quotient constant, and in the other of which—the prism—both are constant; also, a fourth shape was de-

scribed for which the product is constant and the quotient variable. All these cases occur in hydrostatics. The only one not mentioned is that of the figure whose generatrix varies in magnitude as $\frac{a'}{x}$, the first term of last member of eq. (10), while the pressure varies as $a''x$.

Formula (13), or its equivalent, formula (15), in same manner as formula (1), saves the calculator the inconvenience of remembering numerous rules, because the reduction for special cases, a few of which have been illustrated, can be instantly effected before he proceeds to the numerical part of the work. The same may be said of the prismoidal formula, as applied to the summation of these pressures.

The only other submerged figure mentioned in treatises on hydrostatics is the circle, which is usually the surface of a valve. Formula (15) does not apply to this. If r be its radius, and h the depth of centre below surface, then

$$P = h\pi r^2\gamma;$$

plane vertical

and dist. cen. of press. $= h + \frac{r^2}{4h}$.

The statical moment of a material shape, represented by eq. (5), referred to an axis parallel to plane of generatrix, and at a distance d_1 beyond its initial position, is the product of V by the sum of d_1 and expression (1).

$$\therefore \text{Statical moment} = (d_1 + \tfrac{1}{2}D)V + \tfrac{1}{12}D^2(B-A). \qquad (16)$$

This has exactly same advantages as (1).

Formula (16) is very convenient when we would determine the center of gravity, or the statical moment of a compound figure, every part of which is a shape represented by eq. (5). The first term of second member shows that the *c.* of *g.* of each part may be assumed to be half way between its ends; whereafter, by composition, very easily a false statical moment of the whole figure can be obtained. This should be corrected by the second term of second member applied to each part.

The statical moment of the pressure, whose resultant occupies the position indicated by (15), is, referred to intersection of submerged plane and fluid surface, the product of P and expression (15).

To find the moment of the horizontal component of P, referred to fluid surface, which is the moment usually required, multiply the former moment by $(\sin\theta)^2$.

This is useful when we would find the position of center of pressure of a compound figure. Such a figure may be divided into triangles, trapezoids and rectangles, whose bases are parallel to fluid surface; and the moment of each may be found as above.

The moment which is most often required is that of the horizontal component of P, referred to the lower base. This is the product of $\sin\theta$. P and $\sin\theta$. $[d_2 - (15)]$, which is

$$\tfrac{1}{2}D\sin^2\theta.P - \tfrac{1}{12}D^2\gamma\sin^3\theta.(w_2d_2 - w_1d_1), \quad \text{or}$$
$$\tfrac{1}{12}D^2\sin^3\theta.\gamma[2w_1d_1 + (w_1+w_2)(d_1+d_2)]. \qquad (17)$$

This is the moment which tends to overturn the solid whose surface receives the pressure.

Formula (1) will prove useful to the practical engineer, since few shapes to which it is inapplicable come under his consideration. In his service it will be especially simple, for the reason that he will be very likely to have already calculated, for other purposes, the contents of the shapes he deals with, and will, consequently, know at the outset the value of the denominator of second term. For instance, as often occurs, if it be required to find the position of the c. of g. of a piece of iron or timber of known volume or weight and of prismoidal shape, for the purpose of hoisting it, loading it upon a vehicle, or because it is a member of a structure or machine, the formula

$$\frac{(B-A)D^2}{12V},$$

indicating its distance from mid-section toward larger end, is exceedingly simple. If γ be the specific gravity of the material and W the weight of piece, the expression becomes

$$\frac{(B-A)D^2\gamma}{12W}.$$

Even when neither the weight nor volume are known, the practical calculator will find formula (1) very convenient, because it can so easily be reduced to the simplest possible form for special

cases, by mere cancelation of symbols, before the numerical part of the work be commenced.

The calculator can, in most cases, recognize at sight the shapes to which formula (1) applies. For doubtful cases formula (5) is the criterion, and it is easily used. When the shape satisfies this, it is known that both the prismoidal and the center of gravity formulæ apply.

The best practical application of formula (1) is to the determination of the *mean distance*, which the material of a cutting has been hauled to form an embankment, since here it apparently satisfies the greatest want. *Haulage* is the product of the *quantity of material* and the *average haul* or mean distance which it has been transported. *The unit of haulage is one cubic yard hauled one hundred feet.* On this basis the price for haulage is fixed.

After having calculated one factor of haulage, the *quantity*, it remains for us to find the other, or *average haul.* To do this pass a plane anywhere between cut and fill normal to route of haul. Suppose the magnitude of cutting to be generated by a limited plane, whose variable area is represented by y. Let x denote the variable distance of y from the secant plane. However irregular may be the shape of cutting, we know that y varies as a function of x. Hence ydx, the elementary volume, multiplied by x, which produces the elementary amount of haulage, is integrable.

$$\therefore \quad \int_{d_1}^{d_2} xydx,$$

where d_1, d_2, are distances of ends of cutting, is the haulage of cut to plane, and, when x' is the average haul so far as to the plane, the following equation is true.

$$x' = \frac{\int_{d_1}^{d_2} xydx}{\int_{d_1}^{d_2} ydx}. \qquad (18)$$

The first member of this equation is, by definition, the average haul to plane; the second member is, by principles of mechanics, the distance of c. of g. of material from plane, and the equation shows that these are equal.

By similar reasoning it is proven that the average haul from same plane to fill is equal to the distance of c. of g. of fill from plane.

Therefore, *the average haul of a piece of excavation is the distance between the center of gravity of the material as found and its center of gravity as deposited.*

This is as it is stated in works which touch upon the subject. But the practical computation of this theoretical result has been found to be a far more tedious task. It is evident that, first, the centers of gravity of the component solids must be severally ascertained, since the cut or bank is measured as a compound shape. But the application of formula (18) to each of these produces a very intricate expression, involving about double the labor necessary to calculate the true content of the solid by means of the prismoidal formula in crudest shape. After this the several moments must be compounded.

To avoid this some calculators have been in the habit of dividing the excavation into two parts of equal volume by a plane normal to center line, and establishing this as the initial point of the average haul. A plane similarly fixed in the embankment marks the terminal point of same distance. But this plane is always nearer the larger end of shape than the c. of g. is, as may be illustrated upon the cone, triangle or any shape of unequal end dimensions.

For instance, if the first five 100 ft. solids of a railroad cutting have been transported to a bank, and the generatrix commence with an area zero at beginning of cut, and reach, at the end of considered part, an area whose center height is 20 ft., road bed width 20 ft. and side slopes 1½ to 1, the total volume is about 10,000 cubic yards, and, consequently, a difference of 1 ft. in distance makes a difference of one dollar in money. But the difference in the cut is 20 ft., and, if the bank be of same form, 20 ft. is there added to average haul distance. The average haul of other end of same cutting is likely to be also 40 ft. too long. The error of this method, then, makes a total error of eighty dollars for that cutting, which is invariably at the expense of the railroad company. In the time that a calculator would, by this method, compute the haulage of a division of ten miles he would be likely to cost his em

ployers an amount equal to a year's salary.

To divide the cutting into numerous small parts, and find the sum of their moments, or to determine the c. of g. by a mental estimate merely, are methods either laborious or liable to error.

Because the earthwork solid belongs to the class of shapes bounded laterally by straight line surfaces, formula (1) applies thereto. Let A, B, C, ... K, be the areas of cross-sections, a constant distance apart, 100 feet, or D, in a piece of railroad excavation, whose material is all carried in same direction to form an embankment. Assume an axis, outside of cutting, a distance d_1, equal to 50 feet, beyond A. Then, according to formula (16), the statical moment of first volume, referred to this axis, or the haulage of its material to this axis, is, since $d_1+\frac{1}{2}\mathrm{D}=100$,

$$100\mathrm{V}_1+\tfrac{1}{12}\mathrm{D}^2(\mathrm{B}-\mathrm{A}), \qquad (19)$$

The haulage of second volume is, evidently,

$$200\,\mathrm{V}_2+\tfrac{1}{12}\mathrm{D}^2(\mathrm{C}-\mathrm{B}). \qquad (19a)$$

The haulage of each remaining volume is represented by a similar expression, except that the coefficient of first term is always the product of 100, or D, by the ordinal number of the volume. Thus, for last, or n^{th}, volume, the haulage is

$$100\,n\,\mathrm{V}_n+\tfrac{1}{12}\mathrm{D}^2(\mathrm{K}-\mathrm{J}) \qquad (19b)$$

The sum of these expressions is the total haulage to the axis. But the sum of all the second terms is

$$\tfrac{1}{12}\mathrm{D}^2(\mathrm{K}-\mathrm{A}). \qquad (20)$$

Therefore, an exceedingly simple rule for determination of haulage can be constructed.

Before stating this rule let us make a further reduction in the formula. The unit of expressions (19), (20), is a cubic foot hauled a linear foot. To reduce this to the haulage unit, divide by 27D.=2700. Let $\mathrm{V}ol_1=\frac{1}{27}\mathrm{V}_1$, etc. . . . . $\mathrm{V}ol_n=\frac{1}{27}\mathrm{V}_n$, that is, let the abbreviations represent the number of cubic yards instead of cubic feet. This, it happens, is the denomination used in dealing with these quantities, and is, therefore, the denomination in which these quantities are presented to us when we commence to calculate the haulage. Now, divide the sum of expressions (19), (20) by 27D, using the abreviations.

$$\left.\begin{array}{l}\text{Partial}\\ \text{haulage}\end{array}\right\}=\mathrm{V}ol_1+2\mathrm{V}ol_2+\;\ldots\;n\mathrm{V}ol_n$$
$$+\tfrac{100}{324}(\mathrm{K}-\mathrm{A}) \qquad (21)$$

Obviously, the axis may be established anywhere. It is merely convenient to place it midway between the 100 ft. stations. It might occupy the position midway between A and B. Then the moment, or haulage thereto, would be expressed by formula (21), with each coefficient of all but final term [called the *correction term*] decreased by unity. So the first term, $\mathrm{V}ol_1$, would vanish.

Since this is so, the embankment can be referred to the same axis, wherever that may be with respect to the bank. In short, all the terms, except the correction term, express the operation, *to multiply the number of cubic yards in each volume by the number of hundred feet that the mid section of that volume is removed from the axis.*

It is well to place the axis between cut and fill, or, if they overlap, as nearly so as possible, in order to avoid negative moments or haulage. For instance, the axis might be established half way between B and C. Then the partial haulage would be

$$-\mathrm{V}ol_1+\mathrm{V}ol_3+2\mathrm{V}ol_4\;.\;.\;.\;+(n-2)\mathrm{V}ol_n$$
$$+\frac{100}{9\times6\times6}(\mathrm{K}-\mathrm{A}). \qquad (22)$$

This is less than (21), but the difference in defect, 2 $\mathrm{V}ol$, where $\mathrm{V}ol$ is total volume in cubic yards of material removed, is exactly balanced by the same difference in excess, which is created when the haulage from axis to fill is considered.

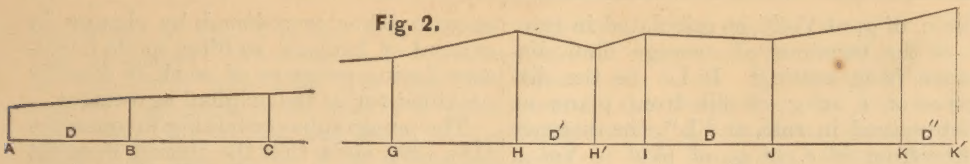
If the irregularity of the ground surface make it requisite to sub-divide one or more of the 100 ft. volumes, as the volume between H and I in Fig. 2, where, also, a plus or intermediate station, K′, instead of a full station, terminates the portion of cutting carried one way; the same rule expressed in italics, second paragraph above, holds, but the correction formula is, for example above,

$$\tfrac{1}{32400}[\mathrm{D}^2(\mathrm{H}+\mathrm{K}-\mathrm{A}-\mathrm{I})+\mathrm{D}'^2(\mathrm{H}'-\mathrm{H})$$
$$+(\mathrm{D}-\mathrm{D}')^2(\mathrm{I}-\mathrm{H}')+\mathrm{D}''^2(\mathrm{K}'-\mathrm{K})], \qquad (23)$$

which is simply a combination of the correction formulæ for the volumes of different lengths.

The haulage from axis to fill is deter-

Fig. 2.



mined in same manner. But the result must be multiplied by

$$\frac{\textit{Excavation Vol.}}{\textit{Embankment Vol.}},$$

for the reason that the material does not occupy same space in embankment as in cut. Ordinary earths become compressed to various degrees. Solid rock fills more space in the bank. The result of division by *Embankment Vol.* is the number of hundred feet from axis to c. of g. of embankment, or the true average haul between. This should be multiplied by *Excavation Vol.*, because the amount of material should be measured in the excavation. *The sum of the haulage from cut to axis and from axis to the fill is the total haulage from cut to fill.*

The foregoing may be condensed into the following systematic

RULE.

To find the haulage of material from a piece of railroad excavation to the embankment built therewith, in terms of the haulage unit, 1 cu. yd. hauled 100 ft.

Consider a plane to be passed midway between two consecutive full stations, as nearly as possible between cut and bank.

Multiply the number of cubic yards in each volume of full [100 *ft.*] *or minor* [*less than* 100 *ft.*] *length, in cutting by the number of hundred feet its mid-section is removed from the plane. If any such mid-section be on the side of plane toward the fill, its product must be taken as negative. The sum of these products is approximately the haulage from the cut to the plane.*

To correct this add the expression

$$100\times\frac{D^2}{9\times 6\times 6}(K-A),$$

once for every series of consecutive volumes of equal length, in the cutting, D *being the length of each such volume,* K, *the area of end cross-section of the series, farther from the plane, and* A, *the area of end cross-section nearer the plane.*

Determine in exactly same manner the haulage from the assumed plane to the fill; but divide result by number of cubic yards in bank, and multiply quotient by number of cubic yards in cutting. The sum of these is the total haulage required.

This rule, although wonderfully simple, in view of what might be expected from so irregular a solid as a railroad cutting, is absolutely correct. It only remains to multiply its result by the price paid per unit of haulage to find the price to be paid for the work.

The position of c. of g. of cut may be determined in same manner described above for the bank.

It frequently occurs that portions of excessive cuttings are transported to spoil-banks near at hand. Often the entire top is taken off by scrapers. The accurate final estimate does not distinguish between these portions. In such a case the haulage from cutting to fill, as found by rule, is not the true haulage. Let *Vol′* be the amount in cubic yards wasted, as recorded in monthly estimates. Let H′ be the haulage of this amount to waste pile, as determined in monthly estimates. The position of c. of g. of this portion of cutting must be known. It can be found, according to method of last paragraph, at the time when that material is measured, and its position should be recorded. Let the distance of this position from the c. of g. of fill be L′. Now, the error in the haulage, as first calculated, is the result of the operation founded on the supposition that *Vol′* was moved to the fill instead of to the waste bank. The correction is, in consequence,

$$H'-L'\times \mathrm{Vol}'. \qquad (24)$$

Quite as often it happens that portions of the embankment are built of material from borrow-pits at hand. Let *Vol″* be the number of cubic yards borrowed, as measured in fill, H″ the haulage thereof, and L″ the distance of c. of g. of this portion of fill from the plane. Let the number of cubic yards in entire part of bank to which material from cut has been hauled be *Vol‴*. This is equivalent with expression, *Embankment Vol*, used above. Then (*Vol‴*−*Vol″*) is the portion of bank brought from cut; and its c. of g.,—not

the c. of g. of Vol''', as calculated in rule, —is the terminus of average haul distance from cutting. If L''' be the distance of c. of g. of fill from plane as determined in rule, and L^{iv}, the distance therefrom of c. of g. of $(Vol'''-Vol'')$, then

$$L^{iv}=\frac{L'''Vol'''-L''Vol''}{Vol'''-Vol''}. \qquad (25)$$

The correction to be made in mean haul distance from cutting is $(L^{iv}-L''')$; and the correction to be made in haulage is, when a portion Vol' of cut, has been wasted,

$$(L^{iv}-L''')\ (Vol-Vol') \qquad (26)$$

$$=\frac{(L'''-L'')Vol''}{Vol'''-Vol''}(Vol-Vol') \qquad (27)$$

When nothing has been wasted, Vol' in (26), (27) is zero.

The haulage H'' from borrow-pit is usually kept separate.

Formulæ (26), (27) are correct on the supposition that the correction (24) has already been made.

It appears from the foregoing that in cases where parts of cuttings are wasted, or parts of embankments are borrowed, not only should the quantities and haulage of such parts be estimated from monthly measurements, as is always done, but also the centers of gravity of these parts should be established and recorded at such times, for use in determination of total haulage finally.

The same methods apply, of course, to the borrow-pits and waste-banks. The method can be readily modified to suit all practical cases.

For another method, also depending upon formula (1), of determining haulage, when the cuts and banks have been calculated entire, that is, when the contents of single volumes are unknown; also, for graphical methods of solving the problems just presented, and for details concerning the terminal solids of the banks and cuts, the reader is referred to the chapter on *Average Haul in Formulæ for Railroad Earthwork, Quantities and Average Haul,* since these depend partially upon formulæ foreign to the nature of this article.

The plan of basing contracts upon excavation and haulage prices, seems to be preferred to that which considers excavation and embankment prices. The advantage of the former is that any difference in labor, occasioned by change in amount of haulage, so often made necessary during progress of work, is directly provided for in the original agreement.

The paragraphs containing expressions (19), (20), show that the statical moment of a series of consecutive, equally long shapes, each of which is represented by some of the forms of a quadratic function, we may find by assuming the c. of g. of each shape to be half way between its ends, then compounding the several moments, and finally correcting by the expression

$$\tfrac{1}{12}D^2(K-A). \qquad (20)$$

To determine the c. of g. of series, divide foregoing result by V. Therefore, find the c. of g. of series on supposition that the c. of g. of each shape is in its mid-section, and correct by adding to the distance, of the point thus found, from A the following:

$$\frac{D^2(K-A)}{12V}. \qquad (28)$$

Thus, formula (1) reduces the problem to this simple one: to find the resultant of a system of parallel forces in one plane, whose intensities and positions are given, the position of this to be corrected by expression (28). The singular advantage of formula (1) is, that its second or correction-term, (28), remains as simple for any number of shapes in the series as for one.

It is evident, in consequence, that the error of the assumption that the c. of g. of each shape is in its mid-section, is comparatively less as the series is longer; also, that no error whatever results from this assumption, when the end areas are equal.

For instance, to find c. of g. of a spherical sector whose component cone and segment have equal altitude, it may be assumed that the c. of g. of each is half way between its bases.

To find c. of g. of a series of trapezoids such as represented in Fig. 2, assume as before the c. of g. of each to be midway its length, and correct the resulting position of c. of g. of series by

$$\frac{D^2(H+K-A-I)+D'^2(H'-H) + (D-D')^2(I-H')+D''^2(K'-K).}{12\,V} \qquad (29)$$

The series of trapezoids may have such arrangement that some are negative, as

illustrated in Fig. 2, p. 291, April No. of this Magazine. When the last ordinate is coincident with first, the algebraic sum of trapezoids is the area of a polygon, included by the top lines of the trapezoids, and the c. of g. or statical moment of this polygon can be found by application of formula (1) or (16).

Suppose the polygon to be ABCDEA. Let a, b, etc., be the ordinates of the vertices; and let a', b', etc., be the corresponding abscissæ of the same. Since we may at first assume the c. of g. of each trapezoid to be midway its length, the several moments of these figures, when the lever arm lies in the direction of abscissæ, are,

for first,

$$\frac{1}{2}(a+b)(b'-a')\frac{1}{2}'(b'+a') = \frac{1}{4}(a+b)(b'^2-a'^2), \qquad (30)$$

for second,

$$\frac{1}{4}(b+c)(c'^2-b'^2), \qquad (30a)$$

. . . .

for last,

$$\frac{1}{4}(e+a)(a'^2-e'^2). \qquad (30b)$$

The correction term, for first trapezoid, is

$$\tfrac{1}{12}(b'-a')^2(b-a); \qquad (31)$$

. . . .

for last,

$$\tfrac{1}{12}(a'-e')^2(a-e). \qquad (31a)$$

The sum of expressions (30), (31), is the statical moment of polygon, when the lever arm is parallel to abscissæ. This may be arranged as follows:

$$\frac{1}{6}[a(b'-e')(e'+a'+b')+b(c'-a')(a'+b'+c')\ldots+e(a'-d')(d'+e'+a')]. \qquad (32)$$

To obtain abscissa of c. of g., divide (32) by area of polygon, as expressed by rule B, p. 293, April No. of this Magazine. Accordingly, distance c. of g. of polygon is

$$\frac{\frac{1}{6}[a(b'-e')(e'+a'+b')+b(c'-a')(a'+b'+c')\ldots+e(a'-d')(d'+e'+a')]}{\frac{1}{2}[a(b'-e')+b(c'-a')\ldots\ldots+e(a'-d')]} \qquad (33)$$

Each term of the statical moment is the product of three factors, two of which are the factors of a term of the area. These two factors need be used once only. This makes the determination easy.

Always, when choice may be had, place the origin at one vertex, as at A. Then $a=o$, $a'=o$. In consequence, one term in the numerator, and one in the denominator, of fraction, vanish, and some of the remaining terms become reduced.

Each term of the statical moment is the continued product of the ordinate of each vertex, the difference between the abscissæ of the two adjacent vertices, (the subtraction being made always in same direction around polygon,) and the sum of the abscissæ of the vertex itself and the two adjacent vertices.

To construct the formula for the ordinate of c. of g., simply change, in formula (33), *a, b, etc., to a', b', etc., and a', b', etc., to a, b, etc.*

Professor Weisbach in article 112 of his *Theoretical Mechanics*, Eckley Coxe's translation, demonstrates a method of determining c. of g. of polygons. The one above presented is, however, decidedly shorter when one co-ordinate only of the c. of g. is required. When both co-ordinates are sought, there is little preference between them, this being in favor of Weisbach's method.

The first two columns of the following example are quoted from the article in *Theoretical Mechanics* above referred to. The problem is solved by the method of this paper.

a'	a	Double Area.	Sextuple Stat. Moment.
24	11	× 11=121	× 49= 5929
7	21	× 40=-840	× 15=12600
-16	15	× 19=285	×-21=-5985
-12	- 9	×-34=306	×-10=-3060
18	-12	×-36=432	× 30=12960
		1984	22444

$$x_1 = \frac{1}{3}\cdot\frac{22444}{1984} = 3.771.$$

To find y_1, substitute a for a', and a' for a, etc.; then proceed as in above example.

Applied to a triangle ABC, when the origin is placed at vertex A, formula (33) becomes

$$\frac{\frac{1}{6}(bc'-cb')\,(b'+c')}{\frac{1}{2}(bc'-cb')} = \frac{\text{Stat. Mom.}}{\text{Area.}}$$

$$\left.\begin{array}{l} =\frac{1}{3}(b'+c')=x_1 \\ \text{Likewise,}\quad \frac{1}{3}(b+c)\ =y_1 \end{array}\right\} \qquad (34)$$

The abscissa and ordinate of c. of g.

of the retaining wall, Fig. 3, the origin being at the left end of lower base, are:

$$x_1 = \frac{d \qquad d+d') + h'(w'-d)(d+d'+w')]}{\frac{1}{2}[hd' + h'(w'-d)]} \tag{35}$$

$$y_1 = \frac{\frac{1}{6}[-dh'(h+h') + d'h(h+h') + w'h'^2)]}{\frac{1}{2}[-dh' + d'h + w'h']} \tag{36}$$

The numerator of (36) can be reduced.

For Fig. 4,

$$x_1 = \frac{\frac{h}{6}[d'(d+d') + (w'-d)(d+d'+w')]}{\frac{h}{2}(w+w')} \tag{37}$$

According to formula (1),

$$y_1 = \tfrac{1}{2}h + \frac{(w-w')h^2}{12V}; \tag{38}$$

or, from (36), when $h=h'$,

$$y_1 = \frac{2w+w'}{w+w'}\,\frac{h}{3} \tag{38a}$$

Fig. 3.

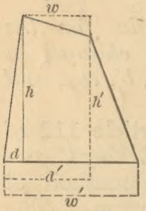


Fig. 4.

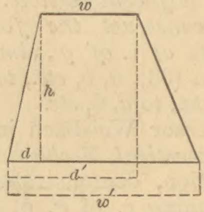
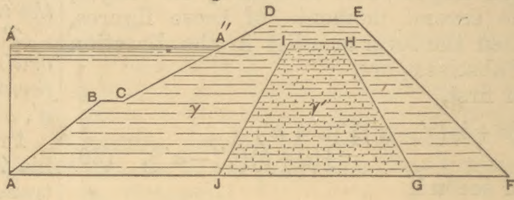


Fig. 5.



To determine the abscissa of the c. of g. of the embankment, Fig. 5, which has a core of different density from that of the covering material, we may find the abscissa of polygon ABCDEFGHIJA, and then that of polygon JIHGJ, and compound the moments, having regard to the unequal densities. But the problem can be solved more simply in one operation. Suppose γ to be the heaviness of covering material, and γ' the heaviness of core material. Now consider at once the polygon ABCDEFGHI JIHGA, regarding the ordinates of I and H, the second time used, as $\frac{\gamma'}{\gamma}i$, $\frac{\gamma'}{\gamma}h$.

To obtain the ordinate of c. of g. of whole dam, consider the same complete polygon, but regard the abscissæ, when used second time, of all the vertices belonging to the core [J, I, H, G], as if they were longer in ratio of γ' to γ.

If it be desired to ascertain the common c. of g. of the water resting on slope, and the dam itself, consider the polygon, AA′A″CBA, BCDEFGHIJ, JI HGA, and regard, when determining the abscissa, the ordinates between first and second commas as multiplied by γ, and those after second comma, as multiplied by γ'. When determining the ordinate of c. of g., multiply the corresponding abscissæ by same quantities.

For the determination of the statical moment, or the c. of g., of any series of trapezoids, formula (32) or (33) is preferable to formula (16) or (1), when the lengths are all different. This is very apt to be the case in that particular arrangement of trapezoids which forms the polygon. (32), (33) are, therefore, eminently suitable to this shape. Whenever a majority of the trapezoids have equal lengths, as in Fig. 2, formula (1) is to be preferred.

As it usually happens that the areas and volumes of practical shapes are calculated for other purposes in advance of the determination of centers of gravity, formula (33) is generally in simplest form. When the area of polygon is neither known nor required, expression (33) can be always reduced by cancelation of common factor $\frac{1}{2}$, and frequently can be much further simplified, as in case of triangle, and to form expressions (37), (38a).

The denominator of (33), when that formula expresses the abscissa, is equal with in value, though different in form from, the denominator of (33) when it expresses the ordinate of c. of g. Therefore, the denominator need be calculated once only. Thus the denominators of (35), (36) are the same in value, each representing the area of the polygon.